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Einstein model for elementary particles

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A group theoretical model unifying a space-time group (E) and an internal symmetry group (I) for strongly interacting particles is worked out. The space-time group is the one that pertains to the group of motions of static Einstein cosmological model implying the symmetry of the group $E = O_4 \otimes R$. With the use of Gueret & Vigier prescription, the left coset $R \otimes O_4$ is identified with the internal symmetry group $I = U_1^B \otimes (SU(2) \otimes SU(2)) \subset SU(4)$. The complete dynamical group (D) is then found to be $D = E \otimes I = (O \otimes R) \otimes U_1^B \otimes (SU(2) \otimes SU(2) \subset SO(4, 2)$. Physically useful representations of the space-time group (E) are worked out by solving the eigenvalue problem of Laplace-Beltmi operator. The internal quantum numbers are prescribed in accordance with the $SU(2) \otimes SU(2)$ model of Nakamura & Sato. A general mass formula is derived and its use for known baryons and mesons is discussed.

1. INTRODUCTION

Recently higher symmetry space-time groups have been used as dynamical groups to obtain mass spectrum of elementary particles. The idea behind the use of such dynamical groups is to reduce a theory in flat space-time with an interaction to a group of motions in curved space; that is, to geometry. In other words, it is to make use of Einstein's idea to describe interaction through the modification of space-time geometry. That this idea is really useful has been shown in a large number of papers (Joseph 1962, Roman & Aghassi 1965a, 1965b, 1966, Prasad 1965, 1967, Barut & Böhm 1965, Böhm 1968, Burcev 1968, Fronsdal 1965) where use has been made of deSitter group as a space-time group.

Further the unitary symmetries $SU(3)$ or $SU(6)$ used in the classification of elementary particles are approximate as the mass spectrum follows only through symmetry breaking machanism (Gell-Mann & Neeman 1964, Gürsey *et al* 1964). This symmetry breaking is introduced in a purely phenomenological manner. As such alternative procedures to derive such mass formulae may be quite relevant. In fact Raczka (1965) derived the Gell-Mann-Okubu mass formula for pseudoscalar mesons by writing the Laplace-Beltrami operator for the manifold $S^2 \times S^1 \times R$ (S- being an n-dimensional sphere, and R a line) which corresponds to the group of motions $O_3 \otimes U_1 \otimes R$.

It is well known that a general Riemannian space does not admit group of motions. Fronsdal (1965) suggested the use of a class of Riemannian spaces which have constant curvature. These spaces do admit group of motions and reduce to Poincare space in the limit of vanishing curvature. de-Sitter space which admits a maximum number of group of motions has been extensively studied. As these higher symmetry groups have to be essentially non-compact (for in the limit they must reduce to Poincare' group), the problem of finding their representations is quite complex. Nevertheless, Rackzka (1966) has shown on physical grounds that the physically useful representations are degenerate representations determined by one or two Casmir invariants. Further he establishes that the most degenerate representations may be constructed by considering Hilbert space $H(x)$ with X a symmetric space of rank one. The Casmir invariant operating on this space is identified with the Laplace-Beltrami operator

$$\Delta(X) = \frac{1}{\sqrt{|g|}} \partial_\mu g^{\mu\nu}(X)\sqrt{|g|}\partial_\nu, \qquad \ldots \quad (1)$$

where $g_{\mu\nu}(x)$ is the Riemannian metric tensor on X and $g = \det g_{\mu\nu}$. The eigenvalues M^2 of $\Delta(x)$ characterize the irreducible representation of the group and the corresponding eigenfunctions span the representation space. Alternatively, we may treat $\Delta(x)$ as generalization of the Klein-Gordon operator $\partial_\mu\partial^\mu$. as in the limit of $g_{\mu\nu}$ taking on Lorentz metric, $\Delta(x)$ reduces to $\partial_\mu\partial^\mu$. Further the operator $\partial_\mu\partial^\mu$ corresponds to one of the Casmir invariants $p_\mu p^\mu$ of the Poincare' group. Therefore, the eigenvalues of $\Delta(x)$ must correspond to the splitting in the mass spectrum obtained from $p_\mu p^\mu$.

Following this idea we have solved the eigenvalue problem of the Laplace-Beltrami operator using the space-time metric as the one corresponding to the static Einstein cosmological universe. It is (Tolman 1934)

$$ds^2 = dt^2-(1-r^2 | K^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad \ldots \quad (2)$$

This gives a mass formula involving mass splitting.

It is known (Tolman 1934) that the metric (2) can be embeded in a pseudo-Euclidean space of 5-dimensions with the signature $(+-\,-\,-\,-)$. The Riemannian manifolds V_4 described by (2) has the geometry of $S^3\times R$, where S^3 describes a hypersphere of radius K given by $K^2 = Z_1{}^2+Z_2{}^2+Z_3{}^2+Z_4{}^2$, and R stands for a line along time axis. We have explicitly worked out its group of motions and have determined the algebra of its infinitesimal generators. It is found to be a seven parameter group $(O_4 \otimes R)$.

We have assumed this to be the external space-time group (E) which contains Poincare' group as its subgroup. In order to determine the corresponding internal symmetry group, we have followed the prescription given by Gueret & Vigier

(1970). Accordingly the internal symmetry group is found to be $I = U_1^{\beta} \otimes (SU(2) \otimes SU(2)) \subset SU(4)$. The unified dynamical group (D) is then found to be $D = E \otimes I = (O_4 \otimes R) \otimes U_1^{\beta} \otimes (SU(2) \otimes SU(2)) \subset SO(4, 2)$.

The internal quantum numbers of elementary particles are labelled in accordance with the $O_4 = SU(2) \otimes SU(2)$ model of Nakamura & Sato (1966, 1968, 1971).

We have next included the contributions from the internal symmetry group (I) in the mass formula obtained for the space-time group (E) and have derived a general mass formula containing the contributions from spin isospin and hyperspin (hypercharge). Applications of this formula for the known baryons and mesons are discussed. Finally it is concluded that this unification of the space-time group (E) and the internal symmetry group (I) is reasonably satisfactory.

2. Solution of the Eigen Value Problem

In order to find the physically useful representations of the group corresponding to Einstein model, we solve the eigen value problem of the following equation :

$$\frac{1}{\sqrt{(-g)}} \partial_\mu [g^{\mu\nu} \sqrt{(-g)} \partial_\nu] \psi - M^2 \psi = 0, \qquad \dots \quad (3)$$

with $g_{\mu\nu}$'s from eq. (2). On substituting these $g_{\mu\nu}$'s from eq. (2) into eq. (3) we obtain the following equations

$$[r^2(1-r^2/K^2)^{\frac{1}{2}} \psi_{,1}]_{,1} + \frac{1}{\sin\theta}(1-r^2/K^2)^{-\frac{1}{2}}(\sin\theta\,\psi_{,2})_{,2} +$$

$$+ \frac{1}{\sin^2\theta}(1-r^2/K^2)^{-1}\psi_{,33} - r^2(1-r^2/K^2)^{-\frac{1}{2}}\psi_{,44} = -r^2(1-r^2/K^2)^{-\frac{1}{2}} M^2 \psi, \quad \dots \quad (4)$$

where $,\mu$ denotes ordinary derivatives with respect to coordinates i.e., $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, $x_4 = t$. Using the usual procedure of separation, namely, writing

$$\psi = R(r)\Theta(\theta)\Phi(\phi)T(t). \qquad \dots \quad (5)$$

We obtain for the angular part the equations

$$\frac{1}{\sin^2\theta}[\sin\theta(\sin\theta\,\Theta_{,2})_{,2}] - m^2\Theta] = -l(l+1)\Theta], \qquad \dots \quad (6)$$

$$\phi_{,33} = -m^2\phi(\phi). \qquad \dots \quad (7)$$

For the time part we have the equation

$$\frac{d^2T}{dt^2} = \epsilon^2 T \qquad \dots \quad (8)$$

which gives a hyperbolic behaviour

$$T = T_0 - \exp(\pm \epsilon t) \quad \ldots \quad (9)$$

for $T(t)$. This behaviour of $T(t)$ agrees wth that of deSitter model (Prasad 1965) for the external space D^+.

On using eqs (6), (7) and (8) in eq (4) we obtain for the radial part the equation

$$r^2(1-r^2/K^2)^{\frac{1}{2}} \frac{d}{dr} \left[r^2(1-r^2/K^2)^{\frac{1}{2}} \frac{dR}{dr} \right] + \left[q^2 - \frac{l(l+1)}{r^2} \right] R = 0, \quad \ldots \quad (10)$$

where

$$q^2 = M^2 - \epsilon^2. \quad \ldots \quad (11)$$

We next substitute

$$\frac{r}{K} = \rho, \quad q^2K^2 = q_p{}^2, \text{ and } x = \left(\frac{\rho^2}{\rho^2-1} \right), \quad \ldots \quad (12)$$

and obtain eq (10) as

$$\left[\frac{d^2}{dx^2} + \frac{3}{2x} \frac{d}{dx} - \frac{q_p{}^2 | 4}{x(x-1)^2} + \frac{l(l+1)/4}{x^2(x-1)} \right] R = 0. \quad \ldots \quad (13)$$

We recognize this as the Papperitz equation (Morse & Feshbach 1953),

$$\frac{d^2\psi}{dz^2} - \left[\frac{\lambda+\lambda'-1}{z} + \frac{\mu+\mu'-1}{z-1} \right] \frac{d\psi}{dz} - \left[\frac{\lambda\lambda'}{z} - \frac{\mu\mu'}{z-1} - \nu\nu' \right] \frac{\psi}{z(z-1)} = 0, \quad \ldots \quad (14)$$

$$\psi = R, \; z = x, \quad \ldots \quad (15a)$$

$$\lambda + \lambda' = -\tfrac{1}{2}, \quad \ldots \quad (15b)$$

$$\mu + \mu' - 1 = 0, \quad \ldots \quad (15c)$$

$$\lambda\lambda' = -\frac{l(l+1)}{4}, \quad \ldots \quad (15d)$$

$$\nu\nu' = 0 \quad \ldots \quad (15e)$$

$$\mu\mu' = -\frac{q_p{}^2}{4}. \quad \ldots \quad (15f)$$

Finally, we substitute

$$\psi = z^\lambda (z-1)^\mu y, \quad \ldots \quad (16)$$

and obtain eq. (14) as

$$z(z-1) \frac{d^2y}{dz^2} + [(2\mu+\lambda-\lambda'+1)z - (\lambda-\lambda'+1)] \frac{dy}{dz}$$
$$+ [\mu(\mu+\lambda-\lambda') - \lambda\lambda'] y = 0. \quad \ldots \quad (17)$$

This is a hypergeometric equation of the type

$$z(z-1)\,\frac{d^2y}{dz^2}+[(a+b+1)z-c]\frac{dy}{dz}+aby=0, \tag{18}$$

with

$$a+b=2\mu+\lambda-\lambda', \qquad \dots \quad (19a)$$

$$ab=\mu(\mu+\lambda-\lambda')-\lambda\lambda', \qquad \dots \quad (17b)$$

$$c=\lambda-\lambda'+1. \qquad \dots \quad (19c)$$

Relations (19) give

$$a=\mu+\lambda, \qquad \dots \quad (20a)$$

$$b=\mu-\lambda'. \qquad \dots \quad (20b)$$

Eq (18) has three singularities namely, at $z=0$, $z=1$ and $z=\infty$.

The solution which is analytic at $z=0$ is the hypergeometric series (Morse & Feshbach 1953)

$$F(a,b\,|\,c\,|\,z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{(z)^n}{n!}, \qquad \dots \quad (21)$$

which converges for $|z|<1$ and is valid for Re $c>0$.

It is well known that all other solutions about other singularities can also be ex pressed in form of eq. (21) with a little adjustment of the parameters.

From eq. (15d), we have

$$\lambda=\frac{l}{2}, \quad \text{and} \quad \lambda'=-\frac{(l+1)}{2}. \qquad \dots \quad (22)$$

Further from eqs. (19) and (22)

$$c=3/2+l$$

Therefore, the condition Re $c>0$ is very well satisfied. Eq. (21) becomes a finite polynomial if either of a and b is a negative integer. We take

$$a=-n, \text{ with } n=0,1,2,3,\dots \qquad \dots \quad (23)$$

Use of eqs. (22) and (23) in eqs. (20) gives

$$\mu=-\left(n+\frac{l}{2}\right). \qquad \dots \quad (24)$$

This implies

$$-\mu=0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \dots$$

From eqs. (15f), (15e) and (12), we got

$$\mu(\mu-1) = \frac{K^2}{4}(M^2-\epsilon^2),$$

The expression on the L.H.S. of this equation is a positive quantity for all values of μ. We write

$$-\mu = S = 0, \tfrac{1}{2}, 1, 3/2, 2, \ldots \qquad \ldots \quad (25)$$

and obtain the above relation as

$$M^2 = \epsilon^2 + \frac{4}{K^2}s(s+1). \qquad \ldots \quad (26)$$

This relation gives the mass formula corresponding to the external group of motions (E). The meanings of the two terms in this relation are made clear in the next section.

3. Group of Motions

In this section we study the group of motions of the manifold that preserves metric (2) and then discuss the resulting algebra of the infinitesimal generators. We begin with the Killing equations (Eisenhart 1926)

$$\xi^\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma} + g_{\mu\sigma}\frac{\partial \xi^\sigma}{\partial x^\nu} + g_{\nu\sigma}\frac{\partial \xi^\sigma}{\partial x^\mu} = 0 \qquad \ldots \quad (27)$$

with $\qquad x'^\mu = x^\mu + \delta x^\mu = x^\mu + \xi^\mu \delta t.$

Using $g_{\mu\nu}$'s from eq. (2) in eq. (27), we obtain the following set of equations :

$$(K_{11}) \quad \frac{\partial \xi^1}{\partial r} = -(1-r^2/K^2)^{-1}\frac{r}{K^2}\xi^1,$$

$$(K_{22}) \quad \frac{\partial \xi^2}{\partial \theta} = -\frac{1}{r}\xi^1,$$

$$(K_{33}) \quad \frac{\partial \xi^3}{\partial \phi} + \cot\theta\,\xi^2 + \frac{1}{r}\xi^1 = 0, \qquad \ldots \quad (28)$$

$$(K_{44}) \quad \frac{\partial \xi^4}{\partial t} = 0,$$

$$(K_{12}) \quad r^2(1-r^2/K^2)\frac{\partial \xi^2}{\partial r} = -\frac{\partial \xi^1}{\partial \theta},$$

$$(K_{13}) \quad \frac{1}{\sin^2\theta}\frac{\partial \xi'}{\partial \phi} = -r^2(1-r^2/K^2)\frac{\partial \xi^3}{\partial r},$$

$$(K_{14}) \quad \frac{\partial \xi^1}{\partial t} = (1-r^2/K^2)\frac{\partial \xi^4}{\partial r},$$

$$(K_{23}) \quad \sin^2\theta \frac{\partial \xi^3}{\partial \theta} = -\frac{\partial \xi^2}{\partial \phi},$$

$$(K_{24}) \quad \frac{\partial \xi^4}{\partial \theta} = r^2 \frac{\partial \xi^2}{\partial t},$$

$$(K_{34}) \quad r^2 \sin^2\theta \frac{\partial \xi^3}{\partial t} = \frac{\partial \xi^4}{\partial \phi}.$$

Use of (K_{22}) in (K_{33}) gives

$$\sin\theta \frac{\partial \xi^2}{\partial \theta} - \cos\theta \xi^2 - \sin\theta \frac{\partial \xi^3}{\partial \phi} = 0. \quad \ldots (29)$$

Further combining (K_{12}) with (K_{13}) and using it in (K_{23}) we get

$$\frac{\partial^2}{\partial r \partial \theta} (\sin\theta \xi^3) = 0. \quad \ldots (30)$$

Eqs. (29) and (30) gives

$$\xi^3 = \frac{f(r, \phi, t)}{\sin\theta} + g(\theta, \phi, t) \quad \ldots (31)$$

and
$$\xi^2 = l(r, \phi, t)\cos\theta + h(\theta, \phi) + k(r, \theta, t). \quad \ldots (32)$$

Further using (K_{24}) in (K_{34}) and cmbining it with (K_{23}) we obtain

$$\frac{\partial^2}{\partial t \partial \theta} (\sin\theta\, \xi^3) = 0.$$

This gives

$$\xi^3 = \frac{p(r, \phi, t)}{\sin\theta} + g(r, \theta, \phi). \quad \ldots (33)$$

Comparing eqs (31) and (33) we have

$$\xi^3 = \frac{f(r,\phi,t)}{\sin\theta} + g(\theta,\phi). \quad \ldots (34)$$

Use of eqs. (32) and (34) in (K_{23}) gives

$$f = \frac{\partial l(r,\phi, t)}{\partial \phi}, \quad \ldots (35)$$

and
$$\frac{\partial h(\theta,\phi)}{\partial \phi} = -\sin^2\theta \frac{\partial g(\theta,\phi)}{\partial \theta}$$

Now using eqs. (34) (35) and (32) in (29) we obtain the following

$$
\begin{aligned}
l &= \omega_1(r, t)\sin\phi + \omega_2(r, t)\cos\phi, \\
f &= \omega_1(r, t)\cos\phi - \omega_2(r, t)\sin\phi, \\
k &= \omega_3(r, t)\sin\theta, \\
g &= \cot\theta(A_1\cos\phi - B_1\sin\phi) + D_1),
\end{aligned} \qquad \cdots \quad (36)
$$

and

$$h = A_1\sin\phi + B_1\cos\phi,$$

where A_1, B_1 and D_1 are constants.

Use of eqs. (32) and (34) in (K_{12}) and K_{24}) along with eq. (36) gives the following

$$\xi^1 = -r^2(1-r^2/K^2)\sin\theta\left[\frac{\partial\omega_1(r, t)}{\partial r}\sin\phi + \frac{\partial\omega_2(r, t)}{\partial r}\cos\phi - \frac{\partial\omega_3(r, t)}{\partial r}\cot\theta\right]$$

$$+f_1(r, t),$$

$$\xi^2 = \cos\theta[\omega_1(r, t)\sin\phi + \omega_2(r, t)\cos\phi)] + \omega_3(r, t)\sin\theta + A_1\sin\phi + B_1\cos\phi,$$

$$\xi^3 = \frac{1}{\sin\theta}[\omega_1(r, t)\cos\phi - \omega_2(r, t)\sin\phi + \cos\theta(A_1\cos\phi - B_1\sin\phi)] + D_1 \quad \cdots \quad (37)$$

and

$$\xi^4 = r^2\sin\theta\left[\sin\phi\,\frac{\partial\omega_1(r,t)}{\partial t} + \cos\phi\,\frac{\partial\omega_2(r, t)}{\partial t}\right] - r^2\cos\theta\,\frac{\partial\omega_3(r, t)}{\partial t} + g_1(r, t).$$

In order to satisfy the remaining equations (K_{11}), (K_{14}), (K_{44}) and (K_{22}) of eq. (28) we substitute (37) in these equations and obtain the following set of equations

$$r\left[\left(2-\frac{3r^2}{K^2}\right)\frac{\partial\omega_i}{\partial r} + r(1-r^2/K^2)\frac{\partial^2\omega_i}{\partial r^2}\right]Y_i = \frac{\partial f_1}{\partial r} + \left(1-\frac{r^2}{K^2}\right)^{-1}\frac{r}{K^2}f_1,$$

$$2r\left[\frac{\partial\omega_i}{\partial t} + r\,\frac{\partial^2\omega_i}{\partial r\partial t}\right]Y_i = -\frac{\partial g_1}{\partial r} - r^2\,\frac{\partial f_1}{\partial r}, \qquad \cdots \quad (38)$$

$$\frac{\partial^2\omega_i}{\partial t^2}\,Y_i = -\frac{\partial g_1}{\partial t},$$

and

$$\left[\omega_i + r^2(1-r^2/K^2)\frac{\partial\omega_i}{\partial r}\right]Y_i = \frac{1}{r}f_1,$$

where $\quad Y_1 = \sin\phi\sin\theta,\ Y_2 = \sin\theta\cos\phi,\ Y_3 = -\cos\theta,$

$i = 1, 2, 3$, and repeated symbols mean summation.

Eqs. (38) can be satisfied if

$$\left(2-\frac{3r^2}{K^2}\right)\frac{\partial\omega_t}{\partial r}+r(1-r^2/K^2)\frac{\partial^2\omega_t}{\partial r^2}=0,$$

$$r\frac{\partial^2\omega_t}{\partial r\partial t}+\frac{\partial\omega_t}{\partial t}=0, \qquad \dots \quad (39)$$

$$\frac{\partial^2\omega_t}{\partial t^2}=0,$$

$$\omega_t+r\left(1-\frac{r^2}{K^2}\right)\frac{\partial\omega_t}{\partial r}=0,$$

and

$$\frac{\partial f_1}{\partial r}+\frac{r}{K^2}(1-r^2/K^2)^{-1}f_1=0, \qquad \dots \quad (40)$$

$$\frac{\partial g_1}{\partial r}+r^2\frac{\partial f_1}{\partial r}=0,$$

$$\frac{\partial g_1}{\partial t}=0,$$

$$f_1=0$$

Eq. (40) gives

$$g_1=g \text{ (a constant), and } f_1=0. \qquad \dots \quad (41)$$

The first and last equations of eqs. (39) are not independent as the first is obtained by differentiating the last. The remainings of eqs. (39) give

$$\omega_t=\omega_t(r),$$

and then the last of eqs. (39) gives

$$\omega_t=c_t\frac{K}{r}(1-r^2/K^2)^{\frac{1}{2}}. \qquad \dots \quad (42)$$

We thus have the following

$$\xi^1=K(1-r^2/K^2)^{\frac{1}{2}}(c_1\sin\theta\sin\phi+c_2\sin\theta\cos\phi-c_3\cos\theta),$$

$$\xi^2=\frac{K}{r}(1-r^2/K^2)^{\frac{1}{2}}(c_1\cos\theta\sin\phi+c_2\cos\theta\cos\phi+c_3\sin\theta)+A_1\sin\phi+B_1\cos\phi \qquad \dots \quad (43)$$

$$\xi^3=\frac{K}{r}(1-r^2/K^2)^{\frac{1}{2}}\operatorname{cosec}\theta(c_1\cos\phi-c_2\sin\phi)+\cot\theta(A_1\cos\phi-B_1\sin\phi)+D_1,$$

and

$$\xi^4=g.$$

There are seven parameters in this group, namely,

$$p^1=A_1,\ p^2=B_1,\ p^3=D_1,\ p^4=C_1,\ p^5=C_2,\ p^6=C_3 \qquad \dots \quad (44)$$

and

$$p^7=g.$$

Next the infinitesimal generators of a Lie group are given by (Eisenhart 1926)

$$X_\sigma = \frac{\partial \xi^k}{\partial p^\alpha} \frac{\partial}{\partial x^k} \qquad (k = 1, 2, 3, 4). \qquad \ldots \quad (45)$$

Using eqs. (43) and (44) in eq. (45) we obtain the following as the infinitesimal generators of this group

$$X_1 = \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi},$$

$$X_2 = \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi}$$

$$X_3 = \frac{\partial}{\partial\phi},$$

$$X_4 = K(1-r^2/K^2)^{\frac{1}{2}}\Big(\sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \sin\phi \frac{\partial}{\partial\theta} + \frac{1}{r}\cos\phi \operatorname{cosec}\theta \frac{\partial}{\partial\phi}\Big),$$

$$X_5 = K\left(1-\frac{r^2}{K^2}\right)^{\frac{1}{2}}\left(\sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi \frac{\partial}{\partial\theta} - \frac{1}{r}\sin\phi \operatorname{cosec}\theta \frac{\partial}{\partial\phi}\right), \qquad \ldots \quad (46)$$

$$X_6 = K(1-r^2/K^2)^{\frac{1}{2}}\left(-\cos\theta \frac{\partial}{\partial r} + \frac{1}{r}\sin\theta \frac{\partial}{\partial\theta}\right)$$

and

$$X_7 = \frac{\partial}{\partial t}.$$

These generators satisfy the following commutation relations

$$[X_i, X_j] = \epsilon_{ijk} X_k, \qquad \ldots \quad (47)$$

with four sets of i, j, k, values, namely, (1, 2, 3), (3, 4, 5), (2, 6, 5) and (1, 6, 4), and

$$[X_1, X_5] = [X_2, X_4] = [X_3, X_6] = 0, \qquad \ldots \quad (48)$$

$$[X_l, X_7] = 0, \text{ with } l = 1, 2, 3, 4, 5, 6. \qquad \ldots \quad (49)$$

The quantum mechanical counter parts $J_{\mu\nu}$, defined in terms of $X_1, X_2, \ldots, X_7$ etc. of these generators are obtained by multiplying these generators by $\pm i(\hbar = 1)$ appropriately. The resulting commutation relations (47) and (48) then satisfy the algebra

$$[J_{\mu\nu}, J_{\alpha\beta}] = i(\delta_{\mu\alpha}J_{\nu\beta} + \delta_{\nu\beta}J_{\mu\alpha} - \delta_{\mu\beta}J_{\nu\alpha} - \delta_{\nu\alpha}J_{\mu\beta}], \qquad \ldots \quad (50)$$

with $\mu, \nu, \alpha, \beta = 1, 2, 3, 4$. This is just the algebra of O_4.

The quantum mechanical counterpart of X_7 is $i(\partial/\partial t)$. This corresponds to translation generator along time axis. The corresponding group is, therefore, R, a line along time axis. The complete group is thus given by $O_4 \otimes R$.

We have already emphasized that the Laplace-Beltrami operator may be identified with the Casmir operator of the group $O_4 \otimes R$. Its irreducible representations expressed in terms of the eigenvalue relation (26) will, therefore, contain contributions both from the Casmir invariant of R as well as that of O_4. As R stands for translation along time-axis, the corresponding translational Casmir invariant $p_\mu p^\mu$ will merely give ϵ^2, where ϵ denotes the rest mass of the particle (m_0), represented by the irreducible representation.

Next the Casmir invariant corresponding to O_4 will give us the eigenvalues equal to $4j(j+1)$ as eq. (47) denotes four sets of rotations. For a free particle it is quite easy to identify j with the spin (s) of the particle. The relation (26) then becomes

$$M^2 = m_0^2 + \frac{4}{K^2} s(s+1). \qquad \dots \quad (51)$$

where the two terms on the r.h.s. are the contributions from the separate Casmir invariants of R and O_4 respectively. The term $1/K^2$ appears because of the metric (2) (Barut & Bohm 1965, Bohm 1968) and is a contribution from curvature. In the limit of $k \to \infty$, metric (2) will tend to Lorentz metric and the splitting due to the second term will disappear. Apart from k the form of (51) is also borne out by our experience of H-atom possessing O_4 symmetry (Pauli 1926, Fock 1935, Bergman 1936).

4. Inclusion of Internal Symmetries—The Complete Model

Following Tolman (1934), we may express metric (2) as

$$ds^2 = (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) - dt^2, \qquad \dots \quad (52)$$

with the substitution

$$z_1 = K(1-r^2/K^2)^{\frac{1}{2}}, \quad z_2 = r \sin\theta \cos\phi,$$
$$z_3 = r \sin\theta \sin\phi, \quad z_4 = r \cos\theta. \qquad \dots \quad (53)$$

The spatial part of the model may, therefore, be interpreted as a three dimensional sphere embedded in a four dimensional Euclidean space (Z_1, Z_2, Z_3, Z_4) with a total spatial volume $= 2\pi^3 K^3$. On including the time part, its complete geometry corresponds to that of a four dimensional cylindrical surface embedded in a five dimensional space. Its group of movements is $O_4 \otimes R$. This we take as our space time group, called the external group (E).

The question of connection between the external group (E) and the corresponding internal symmetry group (I), is very complicated, and as far as we know, no satisfactory solution to this problem has been obtained till this date. Nevertheless, some connection is possible with regard to this model.

We follow the prescription given by Gueret & Vigier (1970) which is closest to our approach. Accordingly, we assume an elementary particle externally to be an Einstein spatial sphere of radius k extended along time axis. Thus in the external event space-time, it appears as a hypercylindrical surface resembling with the hypertube of Gueret & Viger (1970). From the view point of an external observer (ϵ_0), the symmetry admitted by the external event space is that of the group $E = O_4 \otimes R$. The symmetry as seen by the particle observer (ϵ) will be the corresponding internal symmetry admitted by the particle system. This internal symmetry group (I) is given by the left coset of E which is $R \otimes O_4$ in our case. As seen from eq. (49), the translation operaor X_7 commutes with all the generators of O_4. Therefore, Neeman's (1966) objection for the free particles changing their internal quantum numbers during flight is not applicable here.

It is now quite easy to relate this internal symmetry group (I) with the-so called internal symmetries of the elementary particles. For, we may write $O_4 = O_3 \otimes O_3 \sim SU(2) \otimes SU(2)$, and corresponding to R we may take the one parameter gauge group U_1 , both acting on the so called internal space of an elementary particle. Therefore, the full internal symmetry $I = U_1{}^B \otimes (SU(2) \otimes SU(2)) \subset SU(4)$.

The complete dynamical group unifying space-time and internal symmetry groups is then given by

$$D = \quad = O_4 \otimes R \otimes (U_1{}^B \otimes SU(2) \otimes SU(2)) \subset SO(4,2).$$

It may be seen that all the requirements envisaged by Gueret & Vigier (1970) in their scheme of unification are reasonably well satisfied by our model.

Next in order to label the internal quantum numbers of an irreducible representation in terms of I, we make use of the $O_4 = SU(2) \otimes SU(2)$ model of Nakamura & Sato (1966, 1968, 1971) developed as an alternative to $SU(3)$ scheme. Accordingly, one of the $SU(2)$ groups is used to label isospin (τ), as usual, and the other, the hyperspin (ζ), whose third component defines hypercharge Y, through

$$\xi_3 = Y/2. \tag{54}$$

For strong interactions, they require two conservation laws, namely,

$$\begin{aligned} \Delta\,|\tau| &= 0, \\ \Delta\,|\zeta| &= 0, \end{aligned} \tag{55}$$

where the latter represents a new law of ***hypecharg independence.***

In order to construct irreducible representations, they introduce two kinds of spinors t and s as follows

$$t = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad t^\dagger = (\alpha^* \; \beta^*), \qquad \ldots (56)$$

$$s = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad s^\dagger = (\xi^* \eta^*). \qquad \ldots (57)$$

The baryons are assumed to be composed of the bilinear combinations t^*t, s^*s, t^*s, s^*t plus an object F which has baryon number 1 and spin $\frac{1}{2}$. Mesons are then constructed out of B^*B, where B represents any of the baryons.

They define two numbers $\eta(t)$ and $\eta(s)$ as follows

$$\begin{array}{lcccc} & t & t^* & s & s^* \\ \eta(t) = & +1 & -1 & 0 & 0, \\ \eta(s) = & 0 & 0 & +1 & -1. \end{array} \qquad \ldots (58)$$

The ground state baryons and mesons are obtained by imposing the requirement

$$\eta(t)+\eta(s) = 0. \qquad \ldots (59)$$

Higher configurations are then found for the cases

$$\eta(t)+\eta(s) = 2, 4, 6, 8, \ldots.. \qquad \ldots (60)$$

Leptons are also included in this scheme corresponding to

$$\eta(t)+\eta(s) = 1. \qquad \ldots (61)$$

A generalized reflection operator G' is defined in the four dimensional charged space which prescribes a counter multiplet corresponding to every multiplet. We tabulate the internal quantum numbers of the known ground state baryons and mesons as obtained from this scheme in table 1 (Nakamura & Sato 1971).

Following Nakamura & Sato (1971) we consider every strongly interacting particle to be composed of an t, s part and an F part. As far as $t-s$ part is concerned our model is exactly same as that of Nakamura & Sato (1971). For the description of F-part, we make use of U and $O_4 \times R$ groups, the former labelling the baryon number and the latter the spin. Thus, in a way, our model gives complete description of a system involving internal as well as space-time symmetries in a unified manner. In the Nakamura and Sato model F part is included in a purely empirical manner. In this work we have excluded all considerations of leptons.

5. Mass Formula and Mass Spectrum

An irreducible representation of the full dynamical group (D) will now be labelled as $|m_1 s; \tau, \tau_3; \zeta, \zeta_3; B\rangle$, where B denotes baryon number. The symmetry breaking mechanism is already built in our metric (2). This, following Gueret & Vigier (1970) will account for the mass splittings both because of the space-time as well as internal symmetry groups. We have already obtained the mass formula for (E) in eq. (51).

Accordingly, the mass splitting is related to the factor K, where $1/K$ defines the curvature of space. Alternatively, in the language of interactions, this splitting occurs because of some interaction. There are methods to calculate S-matrix and thereby the form of interaction given a metric (Calucci & Ghirardi 1972, Ohta & Okamura 1973). But we postpone such a discussion for a future communication, and assume that the interaction responsible for the mass splitting is some kind of strong interaction mediated by a scaler field. Obviously, k will define the range of this interaction. Moreover, we are working in the units where $h = 1$ and $c = 1$, therefore, the dimensions of mass are $(\text{length})^{-1}$. As such, using the range of strong interaction equal to pion Compton wavelength, we obtain $1/K^2 = m_{0\pi}{}^2$, where $m_{0\pi}$ defines the average mass of pions (= 138 Mev).

Now to include the effect of internal symmetry group in the mass formula we follow the procedure due to Vigier (1969). Accordingly the square mass difference between two states of an irreducible representation is proportional to (μ^2, Ω) where $\mu^2 = J_{\alpha\beta}J^{\alpha\beta}$ and Ω denotes the raising and lowering Weyl-Cartan generator of I.

Denoting our mass operator by M^2, we may write

$$<B; \tau_3,\tau; \zeta_3,\zeta; m,s \,|\, M^2/m_0, s; \zeta, \zeta_3; \tau, \tau_3; B>$$
$$= m^2-m_{0\pi}{}^2 \propto [hs(s+1)+a\zeta(\zeta+1)-b\zeta_3{}^2+c\zeta_3+e\tau(\tau+1)-f\tau_3{}^2+g\tau_3]. \qquad (62)$$

No contribution in this will come from the baryon gauge group because of C-conjugation invariance. As all mass splitting must vanish with $1/K \rightarrow 0$, we take the constant of proportionality to be $1/K^2$. Therefore, our complete mass formula is

$$m^2 = m_0{}^2+m_{0\pi}{}^2\{4s(s+1)+a\zeta(\zeta+1)-b\zeta_3{}^2+c\zeta_3+e\tau(\tau+1)$$
$$-f\tau_3{}^2+g\tau_3\}, \qquad \ldots \quad (63)$$

where $h = 4$ follows from eq. (51).

This is a quadratic mass formula applicable to all the hadrons. Quadratic mass formulae have also been used by several authors (Okubu & Ryan 1964, Roman & Aghassi 1965). Here a, c, c, e, f, g are constant parameters. As illustrative uses of eq. (63) we have tabulated the values of these parameters for baryon multiplets and their counter multiplets in table 2 and for pseudoscalar multiplets and vector meson multiplets in table 3. These values explain the masses of their members (Particle data group 1973). The choice of the mass formulae used has been dictated by the internal quantum numbers of the particles. The various sets of $a, b, \ldots g$ values given in tables 2 and 3 may be understood in terms

Table 1. Internal quantum numbers of particles

	ζ	τ	Representatiton	Particle multiplet	ζ_3	τ_3	Counter multiplet	ζ_3	τ_3	Counter multiplet identified or not
Ground State ·······Baryons·······	$\frac{1}{2}$	$\frac{1}{2}$	$\phi_{\frac{1}{2},\frac{1}{2}}$	p	$\frac{1}{2}$	$\frac{1}{2}$	p'	$\frac{1}{2}$	$\frac{1}{2}$	$N'(1470)$ Identified.
				n	$\frac{1}{2}$	$-\frac{1}{2}$	n'	$\frac{1}{2}$	$-\frac{1}{2}$	
				Ξ^0	$-\frac{1}{2}$	$\frac{1}{2}$	$\Xi^{0\prime}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\Xi'(1820)$ Identified $J^P\xi$
				Ξ^-	$-\frac{1}{2}$	$-\frac{1}{2}$	$\Xi^{-\prime}$	$-\frac{1}{2}$	$-\frac{1}{2}$	
	0	1	$\phi_{1,0}$	Σ^+	0	1	$V^+(Z^+)$	1	0	To be confirmed
				Σ^0	0	0	V^0	0	0	
				Σ^-	0	-1	$V^-(\Omega^-)$	-1	0	Identified. $J^P\xi$
	0	0	$\phi_{0,0}$	Λ^0	0	0	Λ'	0	0	To be confirmed
Ground State Pseudoscalar mesons	0	1	$\phi_{1,0}$	π^+	0	1	D^+	1	0	
				π^0	0	0	$D^0(E(1420))$	0	0	To be confirmed
				π^-	0	-1	D^-	-1	0	
	$\frac{1}{2}$	$\frac{1}{2}$	$\phi_{\frac{1}{2},\frac{1}{2}}$	K^+	$\frac{1}{2}$	$\frac{1}{2}$	$K^{+\prime}$	$\frac{1}{2}$	$\frac{1}{2}$	
				K^0	$\frac{1}{2}$	$-\frac{1}{2}$	$K^{0\prime}$	$\frac{1}{2}$	$-\frac{1}{2}$	To be Identified
				$\overline{K}^0$	$-\frac{1}{2}$	$\frac{1}{2}$	$\overline{K}^{0*}$	$-\frac{1}{2}$	$\frac{1}{2}$	
				K^-	$-\frac{1}{2}$	$-\frac{1}{2}$	$\overline{K}'$	$-\frac{1}{2}$	$-\frac{1}{2}$	
	0	0	$\phi_{0,0}$	η	0	0	$D_0'(X(958))$	0	0	Identified
Ground State Vector mesons	$\frac{1}{2}$		$\phi_{\frac{1}{2},\frac{1}{2}}$	K^{+*}	$\frac{1}{2}$	$\frac{1}{2}$	$K^{+*\prime}$	$\frac{1}{2}$	$\frac{1}{2}$	
				K^{0*}	$\frac{1}{2}$	$-\frac{1}{2}$	$K^{0*\prime}$	$\frac{1}{2}$	$-\frac{1}{2}$	To be confirmed
				$\overline{K}^{0*}$	$-\frac{1}{2}$	$\frac{1}{2}$	$K^{-0*\prime}$	$-\frac{1}{2}$	$\frac{1}{2}$	
				K^{-*}	$-\frac{1}{2}$	$-\frac{1}{2}$	K^{-*}	$-\frac{1}{2}$	$-\frac{1}{2}$	
	0	1	$\phi_{1,0}$	ρ^+	0	1	ϕ^+	1	0	
				ρ^0	0	0	ϕ'	0	0	To be confirmed
				ρ^-	0	-1	ϕ^-	-1	0	
	0	0	$\phi_{0,0}$	ω	0	0	ϕ	0	0	To be confirmed

Table 2. The masses of ground state baryons and their counter multiplets

	Multiplet	a	b	c	e	f	g	Mass formula
Baryon multiplets	(1) $N-\Xi$ ζ-multiplet	-2.225	-2.225	-44.94	0	0	0	$m^2 = m_0^2+m_{0\pi}^2[4s(s+1)+ +a\{\zeta(\zeta+1)-\zeta_3^2\}+c\zeta_3]$
	(2) $n-p$-iso-multiplet	0	0	0	-6	-6	-0.127	$m^2 = m_0^2+m_{\pi 0}^2\,[4s(s+1)+e\{\tau(\tau+1)-\tau_3^2\}+g\tau_3]$
	(3) $\Xi^0-\Xi^-$ -isomultiplet	0	0	0	-6	-6	-0.885	-do-
	(4) $\Sigma^+, \Sigma^0, \Sigma^-$ -isomultiplet	0	0	0	-1.537	-0.113	-0.497	$m^2 = m_0^2+m_{0\pi}^2\,[4s(s+1)+ +e\tau(\tau+1)-f\tau_3^2-g\tau_3]$
Counter Baryon multiplets	(1) $N'-\Xi'$ -ζ-multiplet	-2.784	-2.784	-60.46	0	0	0	$m^2 = m_0^2+m_{0\pi}^2[4s(s+1)+ a\{\zeta(\zeta+1)--\zeta_3^2\}+c\zeta_3]$
	(2) V^+, V^0, V^- -isomultiplet	0	0	0	-0.127	3.964	9.525	$m^2 = m_0^2+m_{0\pi}^2[4s(s+1)+e\tau(\tau+1)-f\tau_3^2+g\tau_3]$

Table 3. For the masses of ground state pseudoscalar mesons and vector mesons

	Multiplet	a	b	c	e	f	g	Mass formula
Pseudoscalar mesons	(1) $\pi^+-\pi^-\pi^0$ -isomultiplet	0	0	0	-0.022	-0.066	0	$m^2 = m_0^2+m_{0\pi}^2[e\tau(\tau+1)-f\tau_3^2]$
	K^+-K^0 -isomultiplet	0	0	0	.00042	.00042	-0.208	$m^2 = m_0^2+m_0^2\,[e\{\tau(\tau+1)-\tau_3^2\}+g\tau_3]$
	$\overline{K}^{-0}-K^-$ -isomultiplet	0	0	0	.00042	.00042	0.208	-do-
Vector mesons	$K^{+*}-K^{0*}$ -isomultiplet	0	0	0	-16	-16	-0.57	$m^2 = m_0^2+m_{0\pi}^2[4s(s+1)+ +e\{\tau(\tau+1)-\tau_3^2\}+g\tau_3]$
	$K^{-*}-K^{-0*}$ -isomultiplet	0	0	0	-16	-16	0.57	

Note—Masses of the counter multiplets of these mesons are not known properly.

of the detailed knowledge of the structures and the dynamics of the particles involved.

For higher spin configurations like those of $J = 3/2$ only one resonance $\Delta_N(3-3)$ and its counter resonance $\Delta_{\Xi}(3-3)$ are predicted (Nakamura & Sato 1971). But only $\Delta_N(3-3)$ is established experimentally with out any knowledge of the mass splitting of its members. As such, no calculation for their masses is possible. But we believe that the mass formula (62) is sufficiently general, and may explain the mass splitting of the higher spin configurations too, once the masses of their members are known experimentally.

6. Conclusion

The model discussed here serves the following purposes.

1. It provides a reasonably satisfactory unification of a space-time group and the corresponding internal symmetry group.
2. The mass splitting within the members of an irreducible representation is automatically built into the structure of the dynamical group (D).
3. No go theorems (Pais 1966) are not applicable to this case as the space-time group involved is not the Poincare' group, but a higher symmetry group.
4. As the space-time group used pertains to a cosmological model, it shows that global structure of the universe provides a possibility of linking the internal symmetries of a particle with its space-time motions. It is suggested that the realistic model of the universe like Friedman model (1922) must be explored for this purpose.

This work needs further investigations about the following:

1. Other solutions of eq. (17) in the form of integral representation along with their analytic continuations are to be investigated (Calucci & Ghirardi 1973).
2. The S-matrix and thereby the nature of scalar interaction built into metric (2) is to be explored.
3. Interpretation of the various sets of $a, b, c, \ldots g$ values is to be found by invoking dynamics of the particles involved.
4. Leptons are to be included in the scheme, and the possibility of a composite model for leptons and baryons (Maki, Ohnuki & Sakata 1965) is to be explored.

Finally, this model has all the merits and demerits of an $SU(2) \otimes SU(2)$ scheme developed as an alternative to $SU(3)$ (Nakamura & Sato 1971)

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